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PP-wave/CFT_2 duality

Jaume Gomis
California Institute of Technology 452-48
Pasadena, CA 91125, USA
E-mail: gomis@theory.caltech.edu

Luboš Motl and Andrew Strominger
Jefferson Physical Laboratory, Harvard University
Cambridge, MA 02138, USA
E-mail: motl@feynman.harvard.edu, andy@planck.harvard.edu

Abstract: We investigate the pp-wave limit of the $AdS_3 \times S^3 \times K3$ compactification of type-IIB string theory from the point of view of the dual $Sym_N(K3)$ CFT. It is proposed that a fundamental string in this pp-wave geometry is dual to the $c = 6$ effective string of the $Sym_N(K3)$ CFT, with the string bits of the latter being composed of twist operators. The massive fundamental string oscillators correspond to certain twisted Virasoro generators in the effective string. It is shown that both the ground states and the genus expansion parameter (at least in the orbifold limit of the CFT) coincide. Surprisingly the latter scales like J^2/N rather than the J^4/N^2 which might have been expected. We demonstrate a leading-order agreement between the pp-wave and CFT particle spectra. For a degenerate special case (one NS 5-brane) an intriguing complete agreement is found.

Keywords: $1/N$ Expansion, Conformal Field Models in String Theory, Penrose limit and pp-wave background, AdS/CFT and dS/CFT Correspondence.

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Contents

1. Introduction

Ever since the realization by 't Hooft that the large-N expansion of gauge theories is organized according to the topology of Feynman diagrams, it has been an outstanding challenge to reorganize gauge theory as some sort of string theory. In a beautiful recent paper, Berenstein, Maldacena and Nastase (BMN) [1] have made progress in this direction with a precise identification between a subset of operators of $\mathcal{N} = 4\,\mathrm{SU}(N)$ super-Yang-Mills and fundamental string excitations [2, 3] in the plane wave geometry [4] that appears in the Penrose limit [1, 5] of $AdS_5 \times S^5$. This correspondence exhibits that strings in this pp-wave are made out of bits, each carrying one unit of longitudinal momentum, and represented by $N \times N$ matrices of the gauge theory.

In this paper we study the Penrose limit of $AdS_3 \times S^3 \times K3$ with the aim of extracting the worldsheet physics of strings in this pp-wave geometry from a particular sector of the dual $Sym_N(K3)$ CFT. As we shall see, the dual CFT — despite not being a gauge theory — captures many features of the string worldsheet and shares many qualitative features, together with interesting differences, with the $\mathcal{N} = 4$ SYM case.

In the lightcone gauge, the string worldsheet in the pp-wave limit of $AdS_3 \times S^3 \times K3$ has four massive directions with mass μ while the other four parametrize the "massless" $K3$ CFT. The stringy spectra on these pp-waves depend on which three-form H fluxes

(R-R or NS-NS) support the geometry. However, the supergravity spectra — with no string oscillator excitations — are identical. We identify a sector of the $Sym_N(K3)$ CFT describing the supergravity spectrum, including the 24 ground states of the massless $K3$ CFT together with the infinite tower of Kaluza-Klein modes on $K3$. The sector of the CFT describing the spectrum of a single string at fixed lightcone momentum p^+ is given by operators in the orbifold $Sym_N(K3)$ CFT having R-charge J in the limit of large N with fixed J^2/N. The relation between J and p^+ is found to be

$$\alpha' p^+ \mu = \frac{J}{g_6\sqrt{N}}\,, \tag{1.1}$$

where g_6 is the six-dimensional string coupling.

The way this agreement comes about is quite illuminating. The (first-quantized) Hilbert space of the $Sym_N(K3)$ CFT in the orbifold limit is (roughly) the same as the second-quantized multi-string Hilbert space of a single $c = 6$ "effective string" in $K3$. The different strings correspond to different permutation cycles (or twisted sectors). The charge J is roughly the length of the cycle, and a single string state in the pp-wave corresponds to a single effective string in the CFT. Hence, when string oscillators are ignored, the fundamental strings and effective strings are the same thing. Further the effective strings can be viewed as chains of "bits" corresponding to the elementary $\mathbb{Z}_2$ permutation, which acts as an $N \times N$ matrix on the cover $(K3)^N$.

This naturally leads to the suspicion that more generally the effective strings *are* fundamental strings, appropriately interpreted in the appropriate limit. At first this seems impossible because the former has a four-dimensional target space while the latter has a (transverse) eight-dimensional target space. However it is proposed that the missing dimensions on the effective string are generated by the action of certain super-virasoro operators. These can also be understood as defects in the string bit picture. This proposal gives the right leading order terms in the oscillator spectrum. We do not give in general an all-orders derivation of the spectrum, which presumably requires analysis of perturbations away from the orbifold point of the $Sym_N(K3)$ CFT. However for a special case (one NS 5-brane) we intriguingly find full agreement (in a sense to be made precise) between the extrapolated spectra. This value of 5-brane charge is actually outside the expected range of validity of the correspondence which uses a supergravity description of the geometry (and requires taking the 5-brane charge to infinity), but nevertheless seems to work. This suggests that this case would be worth understanding better.

Further evidence for the fundamental-effective string identification can be found in a comparison of the genus expansion parameters. The effect of the pp-wave geometry on the strings is to confine them (within a length scale $L = 1/\sqrt{\mu p^+}$) along the massive directions which together with the finite volume $K3$ force the strings to effectively propagate only along the light-cone directions.[1] The effective two-dimensional coupling constant in this

[1] In the pp-wave with a pure NS-NS field strength, the effect of the confining potential is lifted for particular values of p^+. This implies the existence of long strings that can escape all the way to infinity [1, 6].

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case is given by

$$G_N^{(2-\mathrm{dim})} = g_2^2 = \frac{J^2}{N} \tag{1.2}$$

and differs from the one in the $AdS_5 \times S^5$ pp-wave where the two-dimensional effective coupling is $g_2^2 = J^4/N^2$. Therefore, since g_2 is an expansion parameter of string theory in the pp-wave geometry,[2] the duality predicts that the $Sym_N(K3)$ CFT has in the double scaling limit $N \to \infty$, $J \to \infty$ with $J^2/N =$ fixed a consistent and finite effective genus expansion. This expectation fits quite nicely with an elegant paper of Lunin and Mathur [7]. They showed that correlation functions of operators in the twisted sectors of the $Sym_N(K3)$ conformal field theory can be computed by evaluating the path integral over an auxiliary Riemann surface — a covering surface — and that in the large-N limit the correlation function can be organized according to the genus g of the covering surface. (This covering surface is essentially the effective string worldsheet.) Furthermore the path integral over the genus g surface is weighted by N^{-g}. In section 3 we analyze the correlation function involving the class of operators we are interested in, which have a cycle of length $J \to \infty$, and show that in the $N \to \infty$ double-scaling limit with J^2/N finite that the higher genus g contributions are finite, and in fact are weighted precisely by $(J^2/N)^g$. Hence the effective and fundamental string genus expansion parameters are the same (at least in the orbifold limit of $Sym_N(K3)$). Furthermore the power series in J^2/N can be summed up exactly and is found to have an interesting $e^{-J^2/2N}$ behavior.

The rest of the paper is organized as follows. In section 2 we introduce the basic formulae encompassing the well known AdS_3/CFT_2 duality that we will need in the rest of the paper. In section 3 we exhibit the null geodesic and the scaling limit leading to a pp-wave geometry, relate the dual CFT charges with the energy and longitudinal momentum of strings in the pp-wave and exhibit the string spectrum in the pp-wave metric with both NS-NS and R-R flux. In section 4 we identify the chiral primary operators in the relevant sector of the $Sym_N(K3)$ orbifold CFT with the ground states of a string in the pp-wave geometry. Section 5 contains a discussion of string interactions from the dual CFT description. We show that the dual CFT, in the sector dictated by the Penrose limit, exhibits a finite effective genus expansion in J^2/N, precisely matching string theory expectations and explicitly evaluate some correlation functions to all orders in the J^2/N expansion. In section 6 we identify operators in the CFT corresponding to exciting the vacuum states with the zero mode massive oscillators and show how they introduce "impurities" on the operators describing the $p^- = 0$ states. We show that this gives full agreement of the spectrum for the case of one NS 5-brane and more generally gives a leading order agreement. The picture of string bits as elementary twist operators is described in analogy with BMN. We close with some preliminary remarks on the problem of CFT interactions from resolving of the $Sym_N(K3)$ orbifold singularities.

As this work was nearing completion [8] appeared which contains some overlapping observations. See also [9].

[2] The actual physical coupling in a string amplitude carries a factor of g_2 but also depends on the energy of the states involved.

2. Lightning review of AdS_3/CFT_2 duality

The near horizon geometry of the effective string obtained by wrapping Q_5 D5-branes over $K3$,[3] together with Q_1 D1-branes is given by $AdS_3 \times S^3 \times K3$. The metric is

$$ds^2 = R^2(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\phi^2 + d\theta^2 + \cos^2\theta\, d\psi^2 + \sin^2\theta\, d\varphi^2) + ds^2_{K3}\,. \tag{2.1}$$

Near the horizon, the originally arbitrary volume of $K3$ gets fixed in terms of the charges. In this case the volume of $K3$ is given by

$$v \equiv \frac{V(K3)}{(4\pi^2\alpha')^2} = \frac{Q_1}{Q_5}\,. \tag{2.2}$$

R is the AdS_3 and S^3 radius

$$R^2 = \alpha' g_s Q_5 = \alpha' g_6 \sqrt{Q_1 Q_5}\,, \tag{2.3}$$

where g_s is the string coupling constant and g_6 is the effective six-dimensional string coupling constant.

There are many compactifications of the form (2.1) related by U-duality. We will be interested in the S-dual case, consisting of Q_5 NS 5-branes and Q_1 fundamental strings. The metric is then of the form (2.1) but with $K3$ volume

$$v = \frac{g_s^2 Q_1}{Q_5} \tag{2.4}$$

and AdS_3 radius

$$R^2 = \alpha' Q_5 = \alpha' g_6 \sqrt{Q_1 Q_5}\,. \tag{2.5}$$

In this case the effective coupling $g_6^2 = Q_5/Q_1$ is independent of g_s.

Type-IIB string theory on $AdS_3 \times S^3 \times K3$ is dual to a certain two-dimensional $\mathcal{N} = (4,4)$ SCFT living on the cylinder with antiperiodic boundary condition for the fermions. This CFT is a sigma model with resolved symmetric product target space $Sym_N(K3)$, where

$$N = Q_1 Q_5\,. \tag{2.6}$$

The magnitude and form of the resolution of the orbifold singularities depends on the spacetime moduli. This theory can be derived either as the moduli space of cycles (with flat connections) or instantons in $K3$ or as the infrared limit of the Higgs branch of the D1-D5 gauge theory. However in two dimensions (unlike in four), the gauge theory description is not conformally invariant and contains a lot of irrelevant data which disappears when one flows into the infrared. For this reason, it has not been particularly useful in general and we did not find it useful in analyzing the pp-wave limit.

[3] Our analysis applies equally to the T^4 (rather than $K3$) compactification, but for the sake of brevity we restrict our attention to $K3$. The duality is arguably more evident in this case because it is less restricted by symmetries. The generalization to branes wrapping any even cycles in $K3$ is also straightforward.

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3. Penrose limit of $AdS_3 \times S^3 \times K3$

We can now take the Penrose limit of this background by expanding the geometry around a null geodesic near the center AdS_3 orbiting around S^3. We introduce coordinates

$$t = \mu x^+ , \tag{3.1}$$

$$\psi = \mu x^+ - \frac{x^-}{\mu R^2} \tag{3.2}$$

and expand the metric around $\rho = \theta = 0$ by introducing

$$\rho = \frac{r}{R} , \tag{3.3}$$

$$\theta = \frac{y}{R} \tag{3.4}$$

in the $R \to \infty$ limit while keeping α', g_s, g_6 and v (in the R-R as well as NS-NS case) finite, so that we need to take $Q_1, Q_5 \to \infty$ but with Q_1/Q_5 fixed.

The resulting metric is that of a pp-wave

$$ds^2 = -2dx^+dx^- - \mu^2 \left(r^2 + y^2\right) dx^+dx^+ + d\vec{r}_2^{\,2} + d\vec{y}_2^{\,2} + ds^2_{K3} \tag{3.5}$$

supported either by $H^{RR}_{+12} = H^{RR}_{+34} = \mu$ or $H^{NSNS}_{+12} = H^{NSNS}_{+34} = \mu$ and where the volume of $K3$ is given by (2.2) or (2.4), respectively.

We now identify string theory charges in the pp-wave geometry with charges of the dual CFT. By using (3.2) it follows that:

$$p^- = i\partial_{x^+} = \mu(i\partial_t + i\partial_\psi) = \mu(\Delta - J) , \tag{3.6}$$

$$p^+ = i\partial_{x^-} = \frac{-i\partial_\psi}{\mu R^2} = \frac{J}{\mu R^2} . \tag{3.7}$$

Δ is the energy of a state in the CFT in the cylinder or equivalently the conformal dimension of an operator in the complex plane. J is a U(1) subgroup of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ R-symmetry of the $\mathcal{N} = (4, 4)$ SCFT given by $J = J^L_3 + J^R_3$. Note that non negativity of light-cone energy corresponds in the dual CFT description to considering only operators satisfying the BPS bound $\Delta \geq J$. Moreover, vacuum states correspond to chiral primary operators of the CFT.

Considering finite energy excitations in the pp-wave geometry (3.5) and taking into account the $R \to \infty$ scaling limit requires via (3.7) to consider CFT states which have $\Delta, J \to \infty$ as $\sqrt{Q_1 Q_5}$ but have a finite $\Delta - J$.

The string spectrum in the R-R pp-wave background (3.5) has been found in [1, 6] to be

$$\Delta - J = \sum_n N_n \sqrt{1 + \left(\frac{n}{\mu p^+ \alpha'}\right)^2} + \frac{L_0^{K3,v} + \bar{L}_0^{K3,v}}{\mu p^+ \alpha'} , \tag{3.8}$$

where N_n obeys the constraint

$$\sum_{n=-\infty}^{\infty} n N_n = L_0^{K3,v} - \bar{L}_0^{K3,v} \tag{3.9}$$

and is the total occupation number of all worldsheet bosons and fermions with worldsheet momentum n. $L_0^{K3,v}$ is the Virasoro operator for the $K3$ CFT of volume v, where v is given in (2.2). For the NS-NS case the formula is a bit different [1, 6]

$$\Delta - J = \sum_n N_n \left(1 + \frac{n}{\mu p^+ \alpha'}\right) + \frac{L_0^{K3,v} + \bar{L}_0^{K3,v}}{\mu p^+ \alpha'}, \tag{3.10}$$

where v is now given in (2.4). Note however that the supergravity spectrum, where $N_n = 0$ for $n \neq 0$, is identical for both the R-R and NS-NS pp-wave.

4. Matching the ground states

In this section we describe the single-particle chiral primary operators of fixed J in the $Sym_N(K3)$ CFT and show that they match with the expected $p^- = 0$ ground states of a fundamental string in the pp-wave with fixed p^+. This is the first step in identifying the spectra of the two theories.

The general chiral primary operators of the CFT are in one-to-one correspondence with cohomology classes of $Sym_{Q_1Q_5}(K3)$. These can be constructed from the basic cohomology classes of the $K3$ manifold[4] and their counterparts appearing in the Q_1Q_5 twisted sectors of the $S_{Q_1Q_5}$ orbifold CFT. Following [10] the chiral primary states can be written

$$\prod_{i=1}^{M} \alpha_{-n_i}^{A_i} |0\rangle \,. \tag{4.1}$$

The n_i take values from 1 to Q_1Q_5 (labeling the twisted sector) with the restriction that the total length of the string equals Q_1Q_5:

$$\sum_{i=1}^{M} n_i = Q_1 Q_5 \,. \tag{4.2}$$

The index $A = 0, 1, \ldots 23$ runs over the $24 K3$ cohomology classes with $A = 0$ corresponding to the identity. The state

$$\left(\alpha_{-1}^{0}\right)^{Q_1Q_5} |0\rangle \tag{4.3}$$

corresponds to the identity operator on $Sym_{Q_1Q_5}(K3)$ and the dual supergravity vacuum. The number of particles (or strings) in the supergravity dual can be identified with the number of α-oscillators in (4.1) which are *not* α_{-1}^{0}. Hence, enforcing (4.2), the single particle states are of the form

$$\left(\alpha_{-1}^{0}\right)^{M-1} \alpha_{-Q_1Q_5+M-1}^{A} |0\rangle \tag{4.4}$$

with $M \leq Q_1Q_5$.

[4]There is one $(0,0)$ form, one $(0,2)$ form, one $(2,0)$ form, one $(2,2)$ form and twenty $(1,1)$ forms.

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The charges of the general state (4.1) are $(J_3^L, J_3^R) = (\frac{Q_1Q_5 - M + \sum p_i}{2}, \frac{Q_1Q_5 - M + \sum q_i}{2})$, where (p,q) is the Dolbeault cohomology degree of the class associated to the $i^{\rm th}$ oscillator. These charges can be written as a sum over the individual strings:

$$(J_3^L, J_3^R) = \sum_{i=1}^{M} \left(\frac{n_i + p_i - 1}{2}, \frac{n_i + q_i - 1}{2} \right). \tag{4.5}$$

Note that the "nothing-strings" α_{-1}^0 carry vanishing charges.

For single particle states of the form (4.4) the total charge equals

$$J \equiv J_3^L + J_3^R = Q_1 Q_5 - M + \frac{p+q}{2}. \tag{4.6}$$

For any cohomology class, $p+q$ is either $0, 2$ or 4. Hence given any cohomology class, (4.6) can be solved for $M < Q_1Q_5$ as long as J is integer and $J < Q_1Q_5$. Hence within this range there are 24 chiral primaries corresponding to single particle states. Inserting (4.6) in (4.4) these states are

$$\left(\alpha_{-1}^0\right)^{Q_1Q_5 + \frac{p+q}{2} - J - 1} \alpha^A_{-J + \frac{p+q}{2} - 1} |0\rangle. \tag{4.7}$$

These 24 states for each J correspond exactly to the 24 supersymmetric ground states of a fundamental string on the pp-wave geometry (3.5), due to the $K3$ geometry. Hence we see that there is a perfect match between the single-particle chiral primaries and the ground states of a fundamental string. There is a simple explanation of this match. In the orbifold limit of $Sym_N(K3)$, the states are characterized by representations of the permutation group S_N. These may be described as a collection of closed permutation cycles, or $c = 6$ "effective strings" on $K3$ whose length is proportional to the number of elements of the cycle, or equivalently to the order of the twisted sector. This is labelled by the index n_i for the chiral primaries in (4.1). The single particle chiral primaries are those which correspond to a single non-trivial permutation cycle and a single effective string on $K3$. Hence the ground states of the effective string match those of the fundamental string.

The characterization of the states of $Sym_N(K3)$ as a multi-string Hilbert space goes beyond the chiral primaries. This suggests that a fundamental string in a pp-wave in general has a dual description as an effective string in $Sym_N(K3)$! One thing that must be explained is how this is compatible with the fact that the effective string has $c = 6$. We will answer this question in section 6.

5. Interactions

In this section we describe how string interactions in the pp-wave geometry (3.5) are realized in the dual orbifold CFT. We will see that the CFT description of string interactions in (3.5) in the double scaling limit

$$N \longrightarrow \infty \qquad J \longrightarrow \infty, \qquad \text{with } \frac{J^2}{N} \text{ fixed} \tag{5.1}$$

has both similarities and differences with the $\mathcal{N} = 4$ SYM description of interactions of strings in the AdS_5 pp-wave [4].

5.1 Comparison to $\mathrm{U}(N_c)$ gauge theory

In the $\mathcal{N}=4$ $\mathrm{U}(N_c)$ SYM case there is a large N_c (the subscript c is appended to distinguish it from $N = Q_1Q_5$ of the 2D theory) expansion where Feynman graphs are classified according to their topology, so that a genus g graph is weighted by N_c^{2-2g} and planar graphs control the large-N_c limit. However, as shown in [11, 12], when considering free Feynman diagrams involving operators of dimension J, non-planar graphs survive in the double scaling limit (5.1) and free genus g graphs are weighted by $(J^2/N_c)^{2g-2}$. This result meshes quite nicely with the naive expectations of strings in the AdS_5 pp-wave [4]. Massive strings in this geometry are confined in the transverse space and propagate in the light-cone directions, so that the strings are effectively two dimensional. Since the transverse directions are effectively compact, with size $L = 1/\sqrt{\mu p^+}$, the effective two-dimensional string coupling constant written in gauge theory variables is

$$G_N^{(2-\mathrm{dim})} = g_2^2 = g_s^2(\mu p^+\alpha')^4 = \frac{J^4}{N_c^2}\,, \tag{5.2}$$

so that the naive estimate of the weight of a genus g string diagram is just as the result obtained from the free gauge theory $(J^2/N_c)^{2g-2}$.[5]

A similar analysis can be made for the case of strings in the AdS_3 pp-wave (3.5). Massive strings in this pp-wave are also confined in the transverse directions, but now only four directions have a typical scale $L = 1/\sqrt{\mu p^+}$ while the other four directions have a scale determined by the volume of $K3$ at the horizon. Therefore, the effective two-dimensional coupling constant written in the CFT variables is

$$G_N^{(2-\mathrm{dim})} = g_2^2 = g_6^2(\mu p^+\alpha')^2 = \frac{J^2}{N}\,. \tag{5.3}$$

This suggests that correlators in the orbifold CFT must exhibit an expansion in the double scaling limit (5.1) in powers of J^2/N, unlike the $\mathcal{N}=4$ gauge theory in which the expansion is in powers of $(J^2/N)^2$. At first sight there is no obvious reason to expect the two-dimensional CFT to have a well defined $1/N$ expansion. However, we will shortly provide strong evidence for the existence of just such an expansion.

It is interesting that the roles played by N_c in the 4D gauge theory are played by both $N = Q_1Q_5$ and $\sqrt{N} = \sqrt{Q_1Q_5}$ in the 2D CFT. In the 4D case, the maximal charge carried by a chiral primary is $J = N_c$, while in 2D it is $J = N$. The basic string bits of the gauge theory are the $N \times N$ matrices "Z". The role of Z in the 2D gauge theory is played by the lift of the $\mathbb{Z}_2$ chiral primary σ, which acts on the cover of $Sym_N(K3)$ as a matrix permuting the N copies of $K3$ (see section 6.3). The group S_N can also be seen as a discrete relic of the gauge theory group. However these correspondences between N_c and N would lead us to expect that the 2D genus expansion parameter would be $1/N^2$, whereas in fact consistency with (5.3) requires it instead to be $1/N$. The correspondence of N_c with $\sqrt{N}$ might also have been suspected from the D1-D5 gauge theory picture of the 2D CFT, which has a $\mathrm{U}(Q_1)$ gauge group.

[5] However there is a subtlety here in that these estimates do not necessarily reflect the coupling in truly physical amplitudes because the latter have a non-trivial parametric dependence on the energy difference [12].

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5.2 The genus-counting parameter

In the last section we identified the ground states of a fundamental string with those of the effective string. In this section we show that, at least in the orbifold limit of $Sym_N(K3)$, string interactions are described in both cases by a genus expansion. Furthermore the expansion parameters agree as expected from (5.3). Here we use the exact expression for the correlation functions [7, 13, 14] of chiral primary operators to explicitly show that the $Sym_N(K3)$ orbifold CFT, in the sector containing strings of length $J+1$ in the double scaling limit (5.1), has a systematic J^2/N expansion.

In reference [13] the OPE of chiral primary operators in the $Sym_N(K3)$ orbifold CFT were analyzed. The chiral primary operators are constructed from the twist field corresponding to a permutation of length $J+1 \sim J$. The unnormalized two-point function for these chiral primary operators (for the precise form of these operators see (6.10)) is easily evaluated and it is given by

$$\left\langle O_J^\dagger(1)\ O_J(0)\right\rangle = J \cdot \frac{N!}{(N-J)!}\,. \tag{5.4}$$

The large-N expansion of (5.4) for fixed J can be written as

$$\left\langle O_J^\dagger(1)\ O_J(0)\right\rangle = JN^J(1-1/N)(1-2/N)\cdots(1-(J-1)/N)\,, \tag{5.5}$$

so that

$$\left\langle O_J^\dagger(1)\ O_J(0)\right\rangle = JN^J\sum_{g=0}^{\infty}(-1)^g\frac{1}{N^g}\sum_{1\le j_1<j_2<\cdots<j_g\le J} j_1\cdot j_2\ldots j_g\,. \tag{5.6}$$

In order to show that the large-N (genus) expansion is finite in the double scaling limit (5.1) one must show that at a fixed order g in the genus expansion the nested sum in (5.5) scales for large J at most like J^{2g}. For large J, the contribution to the sum is

$$\binom{J}{g}\cdot J^g\,, \tag{5.7}$$

where the first term is the number of terms in the sum and the second one is the (large-J) value of the product of g factors. Therefore, in the large-J limit, where $\binom{J}{g} \simeq J^g$, the leading term scales like J^{2g}, which guarantees that the large-N limit is finite. Therefore, the two-point function (5.4) in the large-N limit can be expressed as a double expansion

$$JN^J\sum_{g=0}^{\infty}\sum_{i=0}^{2g}(-1)^g\frac{1}{N^g}J^i b_{g,i}\,. \tag{5.8}$$

We are interested in the behavior of the two-point function in the double scaling limit (5.1), so that for fixed genus only one term in the sum (5.8) contributes, which yields

$$\left\langle O_J^\dagger(1)\ O_J(0)\right\rangle = JN^J\sum_{g=0}^{\infty}(-1)^g\left(\frac{J^2}{N}\right)^g b_{g,2g} \equiv A_J^{-2}\,. \tag{5.9}$$

This result confirms expectations of the duality and exhibits that the orbifold CFT has a systematic J^2/N expansion. Moreover, one can explicitly evaluate[6] the coefficients $b_{g,2g} = 1/(2^g g!)$ so that the two-point function in the double scaling limit is given by

$$\left\langle O_J^\dagger(1)\ O_J(0)\right\rangle = J N^J e^{-J^2/2N}\,. \tag{5.10}$$

The three point function of chiral primary operators corresponding to cycles of length k, n and $k+n-1$ is given by[7] [13]

$$A_J^3 \left\langle O_{n+k-1}^\dagger(\infty)\ O_k(1)\ O_n(0)\right\rangle = \sqrt{\frac{(n+k-1)^3}{n\ k}}\cdot\sqrt{I}\,, \tag{5.11}$$

where

$$I = \frac{(N-n)!\ (N-k)!}{(N-(n+k-1))!\ N!}\,. \tag{5.12}$$

We now consider the large-N expansion of (5.11) for fixed $n=k=J$. The large-N expansion of I is given by[8]

$$\frac{1}{N}\frac{(1-J/N)(1-(J+1)/N)\cdots(1-(2J-2)/N)}{(1-1/N)(1-2/N)\cdots(1-(J-1)/N)} \tag{5.13}$$

so that

$$I = \frac{1}{N}\sum_{g=0}^{\infty}\sum_{i=0}^{2g}(-1)^g\frac{1}{N^g}J^i\, c_{g,i}\,. \tag{5.14}$$

Therefore, in the double scaling limit (5.1) limit the expansion is finite and is given by

$$I = \frac{1}{N}\sum_{g=0}^{\infty}(-1)^g\left(\frac{J^2}{N}\right)^g c_{g,2g} \tag{5.15}$$

just as we wanted. Moreover, one can resum the series since we can compute the coefficients $c_{g,2g} = 1/g!$. The three point function in the double scaling limit is given by

$$A_J^3 \left\langle O_{2J-1}^\dagger(\infty)\ O_J(1)\ O_J(0)\right\rangle = \sqrt{\frac{8J}{N}}\, e^{-J^2/2N}\,. \tag{5.16}$$

6. Matching the spectrum

6.1 The general R-R case

In this section we consider the general R-R case, with Q_1, Q_5 and J going to infinity in fixed but arbitrary finite ratios. The challenge is to reproduce the spectrum (3.8), which for $\mu p^+\alpha' = J/g_s Q_5$, is given by

$$\Delta - J = \sum_n N_n\sqrt{1+\left(\frac{n g_s Q_5}{J}\right)^2} + g_s Q_5\left(\frac{L_0^{K3,v}+\bar{L}_0^{K3,v}}{J}\right). \tag{6.1}$$

[6]The large-J behaviour of the nested sum (5.6) can be obtained by replacing the sum for an integral.

[7]The three point function is defined by dividing it by the two-point function of the operator being pushed to ∞.

[8]The determination of the highest power of J for given genus g follows in an analogous manner the argument for the two-point function.

We first consider the $K3$ term. In the $Sym_N(K3)$ CFT, the $K3$ appears as the target space of the $Q_1 Q_5$ fractional D1-branes, which arise from each D1-brane fractionating into Q_5 pieces. These have a tension which is smaller than that of a fundamental string by a factor of $g_s Q_5$. (We are in a regime where the radius $g_s Q_5 > 1$.) Hence the $K3$ volume seen by the effective string is $v(g_s Q_5)^{-2}$, so it involves $L_0^{K3,v/g_s^2 Q_5^2}$. This has a simple effect on the zero modes of the effective string, which produce a KK spectrum of modes on $K3$. It simply rescales them: $L_0^{K3,v/g_s^2 Q_5^2} = g_s Q_5 L_0^{K3,v}$. This reproduces for the KK modes the factor of $g_s Q_5$ multiplying the second term in (6.1). The remaining factor of J can be understood as follows. The fields around a twist operator which creates an effective string of length $J+1 \sim J$ at $z=0$ may encircle the origin J times before returning to their original value. Fields are single valued on the J-fold covering space which has a good coordinate $t = z^{1/J}$. It is the CFT on the t-plane which is naturally described by a $c=6$ effective string. L_0^t generates $t\partial_t$ while L_0^z generates $z\partial_z$. Therefore $L_0^t = J L_0^z$. Taking this factor into account the spectrum of KK fluctuations of the effective string exactly matches the corresponding fundamental string excitations contained in the second term of (6.1).

For the non-zero modes, the agreement does not persist, and higher-order corrections are required. We expect that these can be derived along the lines of the computations in BMN in a perturbations expansion about the orbifold limit. We will return to this issue in the last subsection.

Now let's consider the massive excitations, whose spectra are given by the first term in (6.1). These have $\Delta - J$ of order one to leading order in $g_s Q_5/J$. We want to describe these excitations as an operator acting on the chiral primary σ_J (see (6.10) for the precise description of this operator) which creates the charge J effective string. Let us first consider the case $n=0$ with no momentum along the string. This should correspond to an operator with $\Delta - J = 1$. There are four such bosonic operators

$$L_{-1}\,, \qquad \bar{L}_{-1}\,, \qquad J_0^-\,, \qquad \bar{J}_0^-\,, \tag{6.2}$$

as well as their four fermionic superpartners[9]

$$G^-_{-1/2}\,, \qquad \bar{G}^-_{-1/2}\,. \tag{6.3}$$

These eight operators correspond to the zero modes in the massive directions. Massive modes with non-zero worldsheet momentum n correspond to[10]

$$L_{-1-\frac{n}{J}}\,, \qquad \bar{L}_{-1-\frac{n}{J}}\,, \qquad J^-_{-\frac{n}{J}}\,, \qquad \bar{J}^-_{-\frac{n}{J}}\,, \tag{6.4}$$

as well as their fermionic superpartners. These operators are well defined acting on the twist field as long as they are used in combinations with integral $L_0 - \bar{L}_0$. This is the analog of the constraint (3.9). On the t-plane they become integrally-moded

$$L_{-J-n}\,, \qquad \bar{L}_{-J-n}\,, \qquad J^-_{-n}\,, \qquad \bar{J}^-_{-n}\,. \tag{6.5}$$

[9]These supercurrents transform as a doublet under $\mathrm{SU}(2)_F$, the commutant of the tangent space group of $K3$ with the holonomy group, even though we have not explicitly added the label to the operators.

[10]This class of operators were studied in [15], where they were called orbifold Virasoro operators.

To leading order in $1/J$ (with n fixed), the operators (6.4) also have $\Delta - J = 1$. Therfore to this order there is agreement with (6.1). Again an understanding of the $(g_s Q_5 n/J)^2$ correction presumably requires an understanding of perturbations away from the orbifold limit.

A similar discussion yields a leading-order agreement in the general NS-NS case.

6.2 The case of NS-NS $Q_5 = 1$

In this subsection we consider the special case of one NS 5-brane. Strictly speaking, there is no reason to expect the pp-wave duality to remain valid for $Q_5 = 1$, because although J and $N = Q_1 Q_5$ can still be taken to be large, the validity of the Penrose limit as usually described requires the radius of the S^3 (and AdS_3) to be larger than the string scale, which is given by Q_5 (e.g. the worldsheet description of both parts of the worldsheet CFT are given in terms of a level Q_5-2 and Q_5+2 bosonic WZW models, respectively). Nevertheless when the formulae of the pp-wave spectrum are extrapolated (see below) we find agreement.[11] It should be possible, and would be of interest, to compute the large-J spectrum with Q_5 fixed on the fundamental string side directly using the WZW worldsheet CFT (and the recent results of [16]), without making the supergravity approximation and make a comparison to the dual $Sym_{Q_1 Q_5}(K3)$ CFT. However, we should point that the worldsheet description of the superstring on $AdS_3 \times S^3$ breaks down for the case of interest $Q_5 = 1$, since a consistent description requires $Q_5 \geq 2$. On the other hand, there does not seem to be an inconsistency in the WZW description of the pp-wave background obtained in the Penrose limit even for $Q_5 = 1$. For now we content ourselves with the extrapolation of (3.10) to $Q_5 = 1$.

For $Q_5 = 1$, $\mu p^+ \alpha' = J$ and (3.8) becomes

$$\Delta - J = \sum_n N_n \left(1 + \frac{n}{J}\right) + \frac{L_0^{K3,v} + \bar{L}_0^{K3,v}}{J}\,. \tag{6.6}$$

We consider the second term first. This comes from the oscillations of the effective string in the $K3$ of volume v. However the factors of $g_s Q_5$ in the R-R case discussed above (these become factors of Q_5 in the NS-NS case) are now absent. The factor of J can be understood as in that case. Hence we have a match for both zero and non-zero modes. For the massive modes, the dimension of the operators (6.4) exactly matches the $1+n/J$ appearing in (6.6). Hence for this case there is a complete match between the pp-wave and CFT results.[12]

6.3 Twist operators as string bits

The construction of the massive fundamental modes in terms of twisted Virasoro generators in the effective string can be motivated from a string bit picture in analogy to the 4D gauge theory case. This may ultimately be a useful picture for computing corrections and will be the topic of this subsection. We recommend the article [7] as a useful source of facts and formulae about the twist fields on the $\mathcal{N} = (4,4)$ symmetric orbifold CFT.

[11] This is reminiscent of the black hole case, where the entropy-counting works for $Q_5 = 1$ despite the absence of a smooth black hole solution.

[12] This should be understood to be for modes with L_0 or n on the effective string much less than J. If we consider $K3$ modes with L_0 of order J these would include the modes (6.5). Alternatively we could drop both this restriction together with the first term in (6.6).

We begin with the fundamental $\mathbb{Z}_2$ twist operators

$$\sigma^{ij}_{ab}\,, \qquad i \neq j\,, \qquad 1 \leq i,j \leq N\,, \qquad N \equiv Q_1Q_5\,, \qquad a,b = \pm\,, \tag{6.7}$$

which act on the covering space $(K3)^N$. Here i,j determine the two copies of $K3$ which the twist field permutes. The operator is invariant under i-j interchanges. In the orbifold theory, only the operators fully symmetrized with respect to all indices i,j are allowed because of the projection onto S_N invariant states. However we can also construct a product of many operators of the form (6.7) and *then* symmetrize (see below). The operators (6.7) have dimension $(1/2,1/2)$ and the third component of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ R-symmetry is $(a/2,b/2)$; therefore the operator σ^{ij}_{++} is a chiral primary. The $\mathbb{Z}_2$ fields transform in a real representation $(2,2)$ under the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ R-symmetry. Our convention for the normalization of σ^{ij}_{ab} is such that:

$$\sigma^{ij}_{ab}(x,\bar{x})\sigma^{ij}_{cd}(y,\bar{y}) \sim \frac{\epsilon^{ac}\epsilon^{bd}}{(x-y)(\bar{x}-\bar{y})}\,. \tag{6.8}$$

Because of the reality condition for σ^{ij}_{ab}, the contraction with ϵ-symbols can be replaced by Kronecker delta symbols assuming that we conjugate one of σ's in (6.8).

The indices i,j play a role similar to that played by the $\mathrm{SU}(N)$ gauge indices in the 4D gauge theory describing the $AdS_5 \times S^5$ case. The analogy of the specific twist fields with the gauge theory fields is the following (suppressing i,j indices):

$$\sigma_{++} \approx Z\,, \qquad \sigma_{--} \approx \bar{Z}\,, \qquad \sigma_{+-} \approx \Phi\,, \qquad \sigma_{-+} \approx \bar{\Phi}\,. \tag{6.9}$$

In our case, there is only one complex Φ (and its conjugate); the number of the massive directions is reduced to one half of the number in $AdS_5 \times S^5$. Note that σ, like Z is an $N \times N$ matrix. Taking a product of σ fields and symmetrizing is like taking a product of Z's and tracing. S_N acts as a discrete relic of the $\mathrm{U}(N)$ gauge symmetry. However the analogy is not perfect: we have seen that the genus expansion parameter is $1/N$ rather than the $1/N^2$ suggested by the analogy.

The analogy can be pushed further to mimic the strings bit construction of BMN. Z can replaced by σ_{++} and the more general chiral primary analogous to $\mathrm{Tr}\, Z^J$ (the graviton multiplet ground state with a momentum $p^+\mu\alpha' = J/g_6\sqrt{N}$) can be written as

$$O(z) = \frac{1}{\sqrt{N^{J+1}J}} \sum_{i_1,i_2\ldots i_{J+1}}^{1\ldots N} \sigma^{i_1i_2}_{++}(z+\epsilon)\sigma^{i_2i_3}_{++}(z+2\epsilon)\ldots\sigma^{i_Ji_{J+1}}_{++}(z+J\epsilon)\,. \tag{6.10}$$

The sum goes over all i_k different from each other which guarantees the S_N symmetry. The regulator $\epsilon \to 0^+$ is introduced to resolve the ambiguities coming from the branch cuts. We can always consider the branch cuts to be directed in the positive imaginary direction. All the summands come from the $\mathbb{Z}_{J+1}$ twisted sector and $(J_3,\bar{J}_3) = (J/2,J/2)$ for a total $\mathrm{SU}(2)_{R-\mathrm{diag}}$ charge equal to J.

An excited string state can be described for example by replacing one or more of the σ_{++} in (6.10) with one of the "impurities" in (6.9). Apart from $Z,\Phi,\bar{Z},\bar{\Phi}$, we need to know the other impurities analogous to $D_\mu(Z)$ and fermions of the four-dimensional gauge theory. All of them can be obtained from σ_{++} by acting with the following subset of super

Virasoro generators using the standard procedure with contour integrals. These generators (when acting on σ_{++}) create the operators as follows:

Supervirasoro Generator	L_{-1}	$\bar{L}_{-1}$	J_0^-	$\bar{J}_0^-$	$G^-_{-1/2}$	$\bar{G}^-_{-1/2}$
Twist Field	$\sigma_{\oplus +}$	$\sigma_{+\oplus}$	σ_{-+}	σ_{+-}	σ_{0+}	σ_{+0}
$(L_0, \bar{L}_0)$	$(\frac{3}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{3}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(1, \frac{1}{2})$	$(\frac{1}{2}, 1)$
$(J_3, \bar{J}_3)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 0)$

All of them have $\Delta - J \equiv (L_0 + \bar{L}_0) - (J_3 + \bar{J}_3) = 1$. Totally we have four real bosonic impurities and four real fermionic impurities.

At this point the construction of string oscillators and excited string states exactly follows BMN. Summing over the position of the defect weighted by a phase, one can recover for large J the fractional Virasoro generators (6.4) discussed above.

6.4 Away from the orbifold limit

It should be possible to push the string bit analogy further and reproduce the exact spectra (3.8) and (3.10) for all values of the charges along the lines of the analysis given in BMN. One of the difficulties encountered is our incomplete knowledge of the map between the spacetime moduli and the $Sym_N(K3)$ moduli (see [17, 18] for discussion). The derivation of BMN involves the Yang-Mills interactions. In our problem, including interactions means perturbing away from the orbifold limit. In this subsection we offer a few preliminary comments on this problem.

Perturbations away from the orbifold point are generated by supermarginal $\mathbb{Z}_2$ twist fields:

$$m^{ij} \equiv G^a_{-1/2} \bar{G}^b_{-1/2} \sigma^{cd}_{ij} \epsilon^{ac} \epsilon^{bd} \,. \tag{6.11}$$

Note that we contracted the indices $a, b, c, d = \pm$ so that m^{ij} is a singlet $(\mathbf{1}, \mathbf{1})$ under the R-symmetry, i.e. its charges are $(0,0)$. Its dimension is $(L_0, \bar{L}_0) = (1,1)$ because $(1/2, 1/2)$ comes from σ_{ab} and $(1/2, 0) + (0, 1/2)$ comes from the two supercharges. Recall we have omitted the $\mathrm{SU}(2)_F$ indices (see footnote 10), the supercharges $G^-_{-1/2}$ are a doublet under $\mathrm{SU}(2)_F$ and the supercharges $\bar{G}^-_{-1/2}$ are a doublet under the *same* group $\mathrm{SU}(2)_F$. Therefore their tensor product contains a singlet and a triplet:

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3} \,. \tag{6.12}$$

Because the operators m^{ij}_{singlet} and m^{ij}_{triplet} are supermarginal, we can add their symmetrization to the CFT action. The CFT dual to a general $AdS_3 \times S^3 \times K3$ background contains a marginal deformation:[13]

$$S_{\mathbb{Z}_2 \,\text{resolution}} = \lambda \int \mathrm{d}^2 z \sum_{i \neq j}^{1 \ldots N} m^{ij}_{\text{singlet}}(z, \bar{z}) \,. \tag{6.13}$$

[13] We suppress here the possibility of a triplet deformation which arises if the B-field integrated over the three self-dual two-cycles of $K3$ is non zero.

One can treat this operator as a perturbation, analogous to the g_{YM} cubic terms in $\mathcal{N}=4$ gauge theory. In order to generate the correct renormalization of the effective string tension in the R-R case (both for massive and massless part of the fundamental string) as well as the correct coefficient for the matrix string theory-like [19, 20, 21] string interactions [21] we should have $\lambda \sim g_6$. It was indeed argued in [18] that the orbifold point is at $g_6 = 0$. We will offer a heuristic explanation of $\lambda \sim g_6$. D1-branes are able to probe distances of the order of $(T_{D1})^{-1/2} = l_{\rm string}\, g_s^{1/2}$; this determines the typical scale at which the exact orbifold structure of the moduli space is regularized or smeared out. If we turn on the deformation, it blows up a $\mathbb{P}^1$ of area

$$S(\mathbb{P}^1) = \text{coefficient} \times \sqrt{V(K3)}\,. \tag{6.14}$$

Because we expect $S(\mathbb{P}^1)$ to be of order $g_s\alpha'$ and $\sqrt{V(K3)} \approx \alpha' \sqrt{Q_1/Q_5}$ (where the areas are evaluated in spacetime units rather than the units natural for the moduli space), the coefficient must be of order $g_s/\sqrt{Q_1/Q_5} = g_6$. The coefficient should not depend on Q_1/Q_5: because the normalization of the twist fields is T-duality invariant, its coefficient must be also T-duality invariant, i.e. it must be a function of g_6 and $N \equiv Q_1 Q_5$. The coefficient is independent of $Q_1 Q_5$ because of "extensivity" of the conformal field theory in Q_1 or Q_5.

The perturbation theory with the operator (6.13) computed at higher orders encounters the $\mathbb{Z}_3$ twist fields m^{ijk}. The OPE of $\mathbb{Z}_2$ twist fields σ^{ij} and σ^{jk} are generally non zero and contain terms in the $\mathbb{Z}_3$ twisted sectors (because the product of two transpositions is a cyclical permutation with three elements). m^{ijk} can be written as a (normal-ordered) product of $\mathbb{Z}_2$ twist fields. The coefficient g_6^2 of such a $\mathbb{Z}_3$ twist field comes from multiplying two operators of the form (6.13). The $\mathbb{Z}_3$ correction can be viewed as an analogy of the second order contact terms known in Green-Schwarz lightcone gauge string field theory.

We expect that (6.13) plays a similar role to the cubic vertex in gauge theory (which is proportional to g_{YM}) and it generates the fermionic kinetic terms of the form $\theta\theta'$. In a similar fashion, the $\mathbb{Z}_3$ correction will play the role of the quartic vertex (which is proportional to g_{YM}^2) and it is responsible for the bosonic kinetic terms of the form X'^2. Note that in the gauge theory, the cubic vertex trades Z for $\theta\theta$ and therefore it changes the number of fields in the trace by ± 1. In the symmetric orbifold CFT, this rule is replaced by the fact that adding a (non-commuting) transposition from (6.13) into a cyclical permutation changes the length of the cyclical permutation by ± 1. In a similar fashion, adding a non-commuting $\mathbb{Z}_3$ cycle changes the length of a cyclical permutation by 0 or ± 2; the bosonic kinetic terms in the gauge theory also change the number of bosonic impurities by 0 or ± 2. We leave an explicit calculation of these kinetic terms from such perturbations to future work.

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